

## NONSTATIONARY CONCENTRATION POLARIZATION IN LAMINAR ULTRAFILTRATION IN A PLANE CHANNEL

V. I. Baikov and A. V. Bil'dyukevich

UDC 532.542

*A semiintegral method of determination of concentration polarization is suggested and specific features of the latter in laminar ultrafiltration in a plane channel are discussed.*

Stationary concentration polarization in the diffusional boundary layer near the surface of membranes in laminar continuous-flow ultrafiltration has been described mainly by the integral method [1–4]. However it is well known that the results obtained by the integral method depend substantially on the correct specification of the law of the concentration field distribution in the boundary layer. The power [2] or polynomial [3, 4] dependences adopted to describe the concentration field distribution produce a large error or give incorrect results. The last circumstance is due to the fact that the specific features of the boundary conditions over a membrane require that the power and polynomial dependences accurately describe the concentration profile and the diffusional flow simultaneously. This means that prior to solving the equation of convective diffusion it is necessary to have a practically exact solution of it near the membrane surface. The power or polynomial dependences fail to meet this requirement [5].

In the present work a new approach is suggested that may be considered to be a semiintegral method, and it is used to analyze nonstationary concentration polarization in ultrafiltration in a plane slotted channel. This problem is of importance for determination of the time required for continuous-flow ultrafiltration to attain a stationary regime.

For simplicity (without loss of the generality of the method) we dwell on the first stage of laminar ultrafiltration when no gel is formed over the membrane surface during concentration polarization.

Assuming that the membrane possesses ideal selectivity, we consider the nonstationary laminar ultrafiltration process in a plane slotted channel. Then the dimensionless equations of motion, continuity, and convective diffusion acquire the form

$$u \frac{\partial u}{\partial \xi} + v \frac{\partial u}{\partial \eta} = - \frac{\partial P}{\partial \xi} + \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial \eta^2}, \quad (1)$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0, \quad (2)$$

$$\frac{\partial \Theta}{\partial \tau} + u \frac{\partial \Theta}{\partial \xi} + v \frac{\partial \Theta}{\partial \eta} = \frac{1}{\text{Pe}} \frac{\partial^2 \Theta}{\partial \eta^2} \quad (3)$$

with the boundary conditions

$$u = 0, \quad v = -V (\eta = 0); \quad \frac{\partial u}{\partial \eta} = 0 (\eta = 1), \quad (4)$$

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Academic Scientific Complex "A. V. Luikov Heat and Mass Transfer Institute, Academy of Sciences of Belarus," Minsk. Translated from *Inzhenerno-Fizicheskii Zhurnal*, Vol. 67, Nos. 1-2, pp. 103-107, July-August, 1994. Original article submitted March 17, 1993.

$$V\Theta_w + \frac{1}{Pe} \frac{\partial \Theta}{\partial \eta} = 0 \quad (\eta = 0); \quad \frac{\partial \Theta}{\partial \eta} = 0 \quad (\eta = 1), \quad (5)$$

$$\Theta = 1 \quad (\xi = 0). \quad (6)$$

In this statement of the problem the following factor is important. Since  $Pe \gg Re$ , then in considering dynamic problem (1), (2), (4), we may employ the hypothesis of a quasistationary state, i.e., we may assume that each instantaneous concentration distribution corresponds to its stationary velocity distribution at the given moment.

In [6] it is shown that for the dynamic problem with small transmembrane velocities  $V$  and a parabolic velocity profile at the channel inlet we may use, with a high degree of accuracy, the following expressions for the velocity components  $u$  and  $v$ :

$$u = 3(1 - V\xi) \left( \eta - \frac{\eta^2}{2} \right), \quad v = -V \left( 1 - \frac{3}{2}\eta^2 + \frac{\eta^3}{2} \right). \quad (7)$$

Numerical calculations of the concentration polarization in the initial statement (1)-(6) with profile (7) give a deviation not exceeding 7%.

Since the diffusional  $Pe$  number is much greater than unity ( $Pe \simeq 10^7$ ) under real conditions, the thickness of the diffusional layer at any point of the channel is much less than its half-height, i.e., it pertains to the near-membrane region. Then, to solve diffusion problem (3), (5), (6) we may use just the first terms in  $\eta$  in velocity distribution (7):

$$\frac{\partial \Theta}{\partial \tau} + 3(1 - V\xi) \eta \frac{\partial \Theta}{\partial \xi} = \frac{\partial}{\partial \eta} \left( V\Theta + \frac{1}{Pe} \frac{\partial \Theta}{\partial \eta} \right). \quad (8)$$

Next, following the conventional integral method [2-4], we represent the concentration profile as a power or polynomial dependence on  $\eta$ , and we introduce a diffusional layer thickness  $\delta(\tau, \xi)$  that is determined from the condition of satisfying Eq. (8) on the average, i.e., by integrating it with respect to  $\eta$  from 0 to  $\delta$ . The problems arising with such an approach were considered above.

The suggested approach is not associated with a dubious approximation of the concentration profile in the form of a polynomial or power dependence but is based on the concentration distribution obtained directly from Eq. (8). It should be noted that the goal of any calculation of concentration polarization consists in determination of the solute concentration over the membrane surface. We use a method usually adopted in nonstationary problems, when a nonstationary distribution is specified proceeding from the solution of the stationary problem

$$3(1 - V\xi) \eta \frac{\partial \Theta}{\partial \xi} = \frac{\partial}{\partial \eta} \left( V\Theta + \frac{1}{Pe} \frac{\partial \Theta}{\partial \eta} \right). \quad (9)$$

As follows from Eq. (9), in the immediate proximity to the membrane surface, i.e., at  $\eta \rightarrow 0$ , the following expression is valid:

$$\frac{\partial}{\partial \eta} \left( V\Theta + \frac{1}{Pe} \frac{\partial \Theta}{\partial \eta} \right) \approx 0.$$

Double integration and use of the first boundary condition of (5) yields the stationary concentration distribution near the membrane:

$$\Theta = \Theta_w \exp(-Pe V \eta). \quad (10)$$

Proceeding from Eq. (10) and physical considerations based on the concept of a boundary layer, the concentration distribution may be represented by the dependence

$$\Theta_1 = \begin{cases} \Theta_w(\tau, \xi) \exp(-Pe V \eta), & 0 \leq \eta \leq \delta(\tau, \xi), \\ 1, & \delta \leq \eta \leq 1; \end{cases} \quad (11)$$

where the unknown thickness  $\delta$  of the diffusional boundary layer is found from the condition  $1 = \Theta_w \exp(-Pe V \delta)$ , i.e.,

$$\delta(\tau, \xi) = \frac{\ln \Theta_w}{Pe V}. \quad (12)$$

To determine the still unknown concentration  $\Theta_w(\tau, \xi)$  on the membrane, we use the integral equation of mass balance. For this, we will integrate the equation of convective diffusion (3) across the boundary layer from 0 to  $\delta$  and use the first boundary condition of (5) and the condition  $\Theta = 1$  at  $\eta = \delta$ . Then

$$\frac{\partial}{\partial \tau} \int_0^\delta (\Theta - 1) d\eta + \frac{\partial}{\partial \xi} \int_0^\delta u (\Theta - 1) d\eta = V. \quad (13)$$

Substitution of the distributions of velocity (8) and concentration (11) into Eq. (13) gives

$$\begin{aligned} & \frac{\partial}{\partial \tau} \int_0^\delta [\Theta_w \exp(-Pe V \eta) - 1] d\eta + \\ & + \frac{\partial}{\partial \xi} \int_0^\delta 3(1 - V\xi) \eta [\Theta_w \exp(-Pe V \eta) - 1] d\eta = V. \end{aligned}$$

Hence after integration with account for dependence (12) we arrive at

$$\frac{1}{PeV} \frac{\partial}{\partial \tau} (\Theta - \ln \Theta_w - 1) + \frac{3}{(PeV)} \frac{\partial}{\partial \xi} (1 - V\xi) \times \left[ \Theta_w - \ln \Theta_w - \frac{1}{2} (\ln \Theta_w)^2 - 1 \right] = V. \quad (14)$$

From physical considerations, a nonstationary process may be considered as two limiting regime, namely, purely nonstationary and stationary.

We consider the stationary mode of concentration polarization. After integrating with respect to  $\xi$  and applying the boundary condition  $\Theta_w = 1$  at  $\xi = 0$  we obtain from Eq. (14)

$$\Theta_w - \ln \Theta_w - \frac{1}{2} (\ln \Theta_w)^2 = 1 + \frac{V\xi Pe^2 V^2}{3(1 - V\xi)}. \quad (15)$$

The formula obtained agrees well with the numerical solution of the system equations (1)-(6) with parameters typical of the ultrafiltration process ( $Pe V = 10-10^3$ ).

We now analyze relation (15). At small channel lengths when  $\Theta_w \sim 1$ , it may be represented as  $\Theta_w = 1 + \varepsilon$ , where  $\varepsilon < 1$ . Performing a series expansion of  $\ln \Theta_w$  in powers of  $\varepsilon$  and using only cubic terms of the expansion, we obtain

$$\Theta_w = 1 + (2 Pe^2 V^2 V \xi)^{1/3}. \quad (16)$$

When  $\Theta_w \gg \ln \Theta_w$ , we find from (15) the following relation:

$$\Theta_w = \frac{V\xi (PeV)^2}{3(1 - V\xi)}, \quad (17)$$

which is valid either at large  $Pe V$  or at a large channel length. It is noteworthy that formula (15) and its corollaries (16) and (17) give results that practically coincide with the dependences for short, moderately long, and long channels obtained in [7] for reverse osmosis ( $Pe V \simeq 1$ ).

We now analyze the purely nonstationary mode of concentration polarization. After integrating with respect to  $\tau$  and using the initial condition  $\Theta_w = 1$  at  $\tau = 0$  we obtain from Eq. (14)

$$\Theta_{w_1} - \ln \theta_{w_1} - 1 = \text{Pe } VV\tau. \quad (18)$$

Hence for short durations of nonstationary ultrafiltration when  $\Theta_{w_1} \sim 1$ , i.e.,  $\Theta_{w_1} = 1 + \varepsilon$ , and just the initial powers of the expansion of  $\ln \Theta_w$  with respect to  $\varepsilon$  may be used, we have

$$\Theta_{w_1} = 1 + (2 \text{Pe } VV\tau)^{1/2}. \quad (19)$$

For a protracted ultrafiltration process or considerable  $\text{Pe } V$ , when  $\Theta_w \gg \ln \Theta_w$ , we find from (18)

$$\Theta_{w_1} = \text{Pe } VV\tau. \quad (20)$$

Finally, we calculate the time  $\tau_s$  required for the ultrafiltration to attain a stationary state at any point with coordinate  $\xi$ . This time is determined from the condition  $\Theta_w = \Theta_{w_1}$ . Then from relations (16) and (19) as well as (17) and (20), we obtain, respectively,

$$\tau_s = \left( \frac{1}{2} \text{Pe } \xi^2 \right)^{1/3}. \quad (21)$$

$$\tau_s = \frac{\text{Pe } V \xi}{3 (1 - V \xi)}. \quad (22)$$

For illustration purposes, we shall evaluate the time required for the concentration polarization to attain a stationary state in a plane channel with the half-height  $h = 10^{-3}$  m, a length of 1 m, a mean flow rate at the inlet of  $u_0 = 1$  m/sec, the transmembrane velocity  $V = 10^{-5}$  m/sec, and the diffusion coefficient  $D = 10^{-10}$  m<sup>2</sup>/sec. In this case, from formula (22) we have  $t_s = (Vh/3u_0D)X$ , i.e.,  $t_s \sim 30$  sec.

In conclusion, we would like to emphasize once more that the semiintegral method suggested is advantageous since the concentration distribution in the diffusional boundary layer is determined directly from the equation of convective diffusion. In this distribution the unknown parameter is not an arbitrary quantity such as the thickness of the diffusional boundary layer  $\delta$ , which is the determining parameter in a purely integral approach, but the physically clear parameter  $\Theta_w$ , i.e., the solute concentration on the membrane, which characterizes the concentration polarization process in ultrafiltration in many respects.

## NOTATION

$\xi = x/h$ ,  $\eta = y/h$ , dimensionless longitudinal and traverse coordinates;  $u = \hat{u}/u_0$ ;  $v = \hat{v}/u_0$ , dimensionless components of the velocity vector;  $h$ , half-height of the plane channel;  $u_{01}$ , mean velocity at the channel inlet;  $\text{Re} = u_0h/\nu$ , Reynolds number;  $\nu$ , coefficient of kinematic viscosity;  $\text{Pe} = u_0h/D$ , diffusional Peclet number;  $D$ , diffusion coefficient;  $\theta = C/C_0$ , dimensionless concentration of the solute;  $C_0$ , solute concentration at the channel inlet;  $\Theta_w$ , dimensionless concentration of the solute on the membrane;  $V = \hat{V}/u_0$ , transmembrane velocity.

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